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Analytical Solution of Optimal Feedback Control for Radially Accelerated Orbits

Francesco Topputo*
Politecnico di Milano, 20156 Milano, Italy
Ashraf H. Owis†
Cairo University, 12613 Giza, Egypt
and
Franco Bernelli-Zazzera‡
Politecnico di Milano, 20156 Milano, Italy

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The optimal feedback control problem for low-thrust trajectories with modulated inverse-square-distance radial thrust is studied in this paper. The problem is tackled by applying a generating-function method devised for linear systems. Instead of deriving open-loop solutions, arising from the two-point boundary-value problems in which the classical optimal control is stated, this technique allows us to obtain analytical closed-loop control laws. The idea behind this work consists of applying a globally diffeomorphic linearizing transformation that rearranges the original nonlinear dynamic system into a linear system of ordinary differential equations written in new variables. The generating-function technique is then applied to this new dynamic system, the optimal feedback control problem is solved, and the variables are transformed back into the original. Thus, we avoid the problem of expanding the vector field and truncating higher-order terms, because no remainders are lost in the approach undertaken. Practical examples are used to show the usefulness of the derived solution for modulated, inverse-square-distance, radially accelerated orbits.

Nomenclature

A = system state matrix
a_i = coefficients of the final solution ($i = 1, 2$)
B = system control matrix
b_i = coefficients of the final solution ($i = 1, 2, 3$)
c = abbreviation of cos function [$\cos(\theta - \theta_0)$]
c_f = abbreviation of cos function [$\cos(\theta_f - \theta_0)$]
F_2 = generating function
$\mathbf{f}$ = vector field (old variables)
G = nonlinear control matrix (old variables)
H = Hamiltonian
h = angular momentum
J = objective function
L = objective function integrand (old variables)
Q = penalty matrix for the states
R = penalty matrix for the control
r = radius in polar coordinates
s = abbreviation of sin function [$\sin(\theta - \theta_0)$]
s_f = abbreviation of sin function [$\sin(\theta_f - \theta_0)$]
T = objective function integrand (new variables)
t = time, independent variable
$\mathbf{u}$ = control vector (old variables)
$\mathbf{v}$ = control vector (new variables)
v_r = velocity in polar coordinates ($\dot{r}$)
$\mathbf{x}$ = state vector (old variables)
$\mathbf{y}$ = state vector (new variables)
$\boldsymbol{\alpha}$ = additive map for control
β = multiplicative map for control
$\Delta\theta$ = difference anomaly ($\theta - \theta_0$)
θ = anomaly in polar coordinates
$\boldsymbol{\lambda}$ = vector of costates or Lagrange multipliers
μ = gravitational constant
τ = new independent variable

Subscripts

xx = submatrix associated with x^2 terms
$x\lambda$ = submatrix associated with $x\lambda$ terms
$\lambda\lambda$ = submatrix associated with λ^2 terms
$0, f$ = initial, final

I. Introduction

THE advantages of low-thrust propulsion applied to steer spacecraft have recently been demonstrated by two missions: NASA's Deep Space 1 and ESA's SMART-1. Using mass expulsion systems, the high specific impulse associated with the low-thrust engine entails a sensible reduction of the propellant mass fraction; on the other hand, when unconventional systems are considered, such as solar sails or minimagnetospheric plasma propulsion, no propellant is required. In any case, the final outcome is a reduced mass at launch or an increased payload mass.

Although low-thrust propulsion gives rise to advantages from the total mass standpoint, the trajectory design for spacecraft equipped with these systems becomes less trivial than that associated with spacecraft propelled by chemical propulsion. In fact, chemical propulsion is usually assumed to produce instantaneous velocity changes, whereas low thrust acts for a long time during the transfer and needs more refined mathematical tools when dealing with it. One of these tools is the optimal control theory used to find solutions that both minimize performance index and satisfy the mission constraints.

Historically, optimal low-thrust transfers have been first tackled with indirect and then with direct methods. The former stems from Pontryagin's maximum principle using the calculus of variations

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*Postdoctoral Fellow, Aerospace Engineering Department, Via La Masa, 34.

†Lecturer, Astronomy and Meteorology Department, Cairo University Street.

‡Full Professor, Aerospace Engineering Department, Via La Masa, 34. Senior Member AIAA.

[1,2]; the latter aims at solving the problem via a standard nonlinear programming procedure [3]. Even though it can be demonstrated that one approach is the approximation of the other [4,5], the direct and indirect methods have both different advantages and drawbacks; they require the solution of a complex set of equations: the Euler–Lagrange differential equations for the indirect methods and the Karush–Kuhn–Tucker algebraic equations for the direct methods [5].

The guidance law designed with these methods is obtained in an open-loop context; the nominal control history, even if minimizing the prescribed performance index, is not able to respond to any perturbation that could alter the state of the spacecraft. Thus, if the initial conditions are slightly varied (e.g., due to small launch errors), the optimal solution needs to be computed again. The outcome of the optimal control problem is, in fact, an optimal guidance law expressed as a function of time, $\mathbf{u} = \mathbf{u}(t)$ $(t \in [t_0, t_f])$, where t_0 and t_f are the initial and final times of the controlled phase, respectively, and $\mathbf{u}$ is the control vector.

Instead of the classic optimal guidance, in this paper, both the optimal guidance and control are studied and applied to low-thrust trajectories. This is referred to as the optimal feedback control problem. With this approach, the solution that minimizes the performance index is also a function of the current state $\mathbf{x}$; the outcome is, in fact, an optimal guidance law $\mathbf{u} = \mathbf{u}(\mathbf{x}, t)$ $(t \in [t_0, t_f])$. This represents a closed-loop solution. If the current state is perturbed, $\tilde{\mathbf{x}} = \mathbf{x} + \delta\mathbf{x}$, we are able to compute the new optimal solution by simply evaluating $\mathbf{u} = \mathbf{u}(\tilde{\mathbf{x}}, t)$ $(t \in [t_0, t_f])$, thus avoiding the solution of another optimal control problem. This property holds by virtue of the analyticity of the control law (the proposed approach solves the optimal feedback control problem in a totally analytical fashion) and only requires the accessibility of the current state. Moreover, when the current state is set to the initial condition, it is possible to extract the nominal guidance law that solves the classical optimal control problem.

The optimal feedback control for linear systems with quadratic objective functions is addressed through the matrix Riccati equation [2]: a matrix differential equation that can be integrated backward in time to yield the initial value of the Lagrange multipliers. The same problem was tackled in an elegant fashion using the Hamiltonian dynamics and exploiting the properties of the generating functions [6,7]. With this approach, it is possible to devise suitable canonical transformations (satisfying the Hamilton–Jacobi equation) that also verify both the two-point boundary-value problem associated with Pontryagin's principle and the Hamilton–Jacobi–Bellman equation of the optimal feedback control problem. The generating-function method was extended to nonlinear dynamic systems supplemented by quadratic objective functions; in this case, the vector field is expanded in Taylor series and the optimal control is derived as a polynomial [6]. Nevertheless, the resulting optimal control differs from that obtained through application of Pontryagin's principle, because in the process of series expansion and truncation, the dynamics associated with the high-order terms is neglected. Recently, the nonlinear feedback control of low-thrust orbital transfers has been analyzed using continuous orbital elements feedback and Lyapunov functions [8].

In this work, the optimal feedback control problem is solved in the frame of a nonlinear vector field, the two-body dynamics, supported by a nonlinear objective function. The idea consists of applying a globally diffeomorphic linearizing transformation that rearranges the original problem into a linear system of ordinary differential equations and a quadratic objective function written in a new set of variables [9]. The generating-function technique is then applied to this new problem, and the optimal feedback guidance is derived and transformed back as a function of the original variables. In this way, we thus avoid the series expansion and truncation process, because no information related to high-order terms is lost.

The dynamics of the two-body motion with radial thrust are considered. The dynamics of low-thrust propulsion with constant radial thrust have widely been studied by a number of authors [10–13]. In this paper, we consider a radial acceleration that varies according to the inverse-square distance from the sun and for which the magnitude can be modulated. This kind of thrust can be generated by both sun-facing solar sails and minimagnetospheric plasma propulsion, which are able to modulate the thrust magnitude [14]. In principle, this radial thrust can be also associated with the solar electric propulsion. Even in this case, in fact, the power supplied to the electric engine, generated by solar arrays, decreases with the inverse-square distance from the sun. In addition, the thrust magnitude can be modulated by tuning the propellant mass flow [15,16].

McInnes [14] studied the families of orbits generated by these dynamics (a generalized two-body problem) with both forward propagation and inverse approach. These solutions can be obtained by quadrature because the equations of motion reduce to a linear differential equation in new variables after a suitable coordinate change (this is possible because the angular momentum is conserved during the motion). The considered radial thrust indeed changes the spacecraft's energy, but conserves its angular momentum; therefore, only transfers between orbits with the same angular momentum are possible (i.e., transfers between circular orbits are forbidden). Yamakawa [17] introduced the gravity-assist maneuvers in these radially accelerated orbits to derive escape trajectories from the solar system or transfers between circular orbits. Nevertheless, previous studies did not deal with the optimal feedback control of these low-thrust trajectories.

The remainder of the paper is organized as follows. In Sec. II, the dynamic system is presented and the optimal control problem is stated. In Sec. III, the principles of the linearizing transformations are briefly discussed and then applied to the stated problem; the outcome is a linear dynamic system and a quadratic objective function written in new variables. In Sec. IV, the new problem is stated as a linear-quadratic regulator and its solution through the generating-function method is recalled. In Sec. V, the linear problem is solved and the result is transformed back into the original variables. The analytic solution of optimal feedback low-thrust trajectories is discussed by means of sample cases. Final remarks and possible future applications are pointed out in Sec. VI.

II. Statement of the Problem

The motion of a spacecraft is considered under the influence of the gravitational attraction of one central body, the sun in our case, along its entire orbit. In addition, the following assumptions are made: the motion is planar (i.e., it can be described with two degrees of freedom) and the low thrust is radial, proportional to the inverse-square distance from the central body, and can be modulated in magnitude. The equations of motion in polar coordinates (r, θ) read

$$\ddot{r} - r\dot{\theta}^2 + \frac{\mu}{r^2} = \frac{u}{r^2}, \qquad r\ddot{\theta} + 2\dot{r}\,\dot{\theta} = 0 \tag{1}$$

where μ is the gravitational constant of the sun, and $u(t)$ $(u\colon \mathbb{R} \to \mathbb{R})$ is is the control used to modulate the magnitude of the radial acceleration. The second half of Eqs. (1) can be rewritten as

$$\frac{1}{r}\,\frac{\mathrm{d}}{\mathrm{d}t}(r^2\dot{\theta}) = 0 \tag{2}$$

meaning that the specific angular momentum $h = r^2\dot{\theta}$ is conserved during the motion; therefore, the orbits lie on the manifold

$$\mathcal{H} = \{(r, \theta, \dot{r}, \dot{\theta}) \in \mathbb{R}^4 | h = \text{const}\}$$

This condition can be used to lower the order of Eqs. (1), yielding

$$\ddot{r} + \frac{\mu}{r^2} - \frac{h^2}{r^3} = \frac{u}{r^2}, \qquad \dot{\theta} = \frac{h}{r^2} \tag{3}$$

The system of equations (3) can be rearranged into three first-order equations

$$\dot{r} = v_r, \qquad \dot{\theta} = \frac{h}{r^2}, \qquad \dot{v}_r = \frac{h^2}{r^3} - \frac{\mu}{r^2} + \frac{u}{r^2} \tag{4}$$

and rewritten in the compact form

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) + G(\mathbf{x})u \tag{5}$$

with

$$\mathbf{x} = [r, \theta, v_r]^T, \quad \mathbf{f}(\mathbf{x}) = \left[v_r, \frac{h}{r^2}, \frac{h^2}{r^3} - \frac{\mu}{r^2}\right]^T, \quad G(\mathbf{x}) = \left[0, 0, \frac{1}{r^2}\right]^T \tag{6}$$

where the total vector field was purposely separated into two terms to match the conditions of applicability of linearizing maps (see the next section).

Assume now that the following performance index must be minimized:

$$J = \frac{1}{h}\int_{t_0}^{t_f} \frac{u^2}{r^2}\,\mathrm{d}t \tag{7}$$

where t_0 and t_f are the initial and the final times, respectively. Performance index (7) is slightly different from the standard quadratic-control objective function used in space trajectory optimization [6]. The weighing factor $1/r^2$ is introduced to obtain a quadratic objective function, and therefore a neat analytical solution, once the problem is reformulated using the new variables. This choice reflects the scope of the paper that mostly aims at demonstrating the feasibility of the undertaken approach to solve feedback control problems with nonlinear systems, rather than performing a standard trajectory optimization.

The optimal control problem is stated by means of dynamic system (5), objective function (7), and the following fixed-state two-point boundary conditions

$$\begin{cases} r(t_0) = r_0 \\ \theta(t_0) = \theta_0 \\ v_r(t_0) = v_{r,0} \end{cases} \qquad \begin{cases} r(t_f) = r_f \\ \theta(t_f) = \theta_f \\ v_r(t_f) = v_{r,f} \end{cases} \tag{8}$$

with fixed t_0 and t_f.

III. Linearizing Maps for Nonlinear Dynamical Systems

In this section, the problem stated through Eqs. (5–8) is transformed into a new problem, written using different variables, in which the equations of motion turn out to be linear. In general, this transformation can be applied to a class of nonlinear systems for which the dynamics can be written as [9]

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) + G(\mathbf{x})\mathbf{u} \tag{9}$$

where $\mathbf{x} \in \mathbb{R}^n$, $\mathbf{u} \in \mathbb{R}^m$, $\mathbf{f}$: $\mathbb{R}^n \to \mathbb{R}^n$, and G is a $n \times m$ matrix for which the elements $g_{ij}(\mathbf{x})$ are g_{ij}: $\mathbb{R}^n \to \mathbb{R}$, where $i = 1, \ldots, n$; $j = 1, \ldots, m$; and n is a multiple integer of m (i.e., $n = pm$ and $p \in \mathbb{N}^+$). The objective function is assumed to be

$$J = \int_{t_0}^{t_f} L(\mathbf{x}, \mathbf{u})\,\mathrm{d}t \tag{10}$$

where $L(\mathbf{x}, \mathbf{u})$: $\mathbb{R}^{n+m} \to \mathbb{R}$ is a generic nonlinear function of the states and the controls, and t_0 and t_f are fixed. Finally, both the initial and final states are assumed to be given [i.e., $\mathbf{x}(t_0) = \mathbf{x}_0$ and $\mathbf{x}(t_f) = \mathbf{x}_f$]. Following the procedure described in [9], we search for a globally diffeomorphic linearizing transformation

$$\mathbf{y} = M(\mathbf{x}) \tag{11}$$

$$\mathbf{u} = \alpha(\mathbf{x}) + \beta(\mathbf{x})\mathbf{v} \tag{12}$$

such that the new state-space representation of dynamic system (9) becomes

$$\mathbf{y}' = A\mathbf{y} + B\mathbf{v} \tag{13}$$

where $\mathbf{y}' = \mathrm{d}\mathbf{y}/\mathrm{d}\tau$ (τ is the new independent variable); A and B are $n \times n$ and $n \times m$ constant matrices, respectively; M: $\mathbb{R}^n \to \mathbb{R}^n$; α: $\mathbb{R}^n \to \mathbb{R}^m$; β is a $m \times m$ matrix for which the elements $\beta_{ij}(\mathbf{x})$ are β_{ij}: $\mathbb{R}^n \to \mathbb{R}$ ($i, j = 1, \ldots, m$); and $\mathbf{v} \in \mathbb{R}^m$. Maps (11) and (12) can be applied to dynamic system (9) to produce new linear state-space representation (13). Furthermore, a new objective function is also obtained by applying transformations (11) and (12) to Eq. (10).

The derivative $\mathbf{y}'$ can be written as

$$\mathbf{y}' = \frac{\partial M \partial \mathbf{x}\,\mathrm{d}t}{\partial \mathbf{x}\;\partial t\,\mathrm{d}\tau} = \frac{\partial M}{\partial \mathbf{x}}(\mathbf{f} + G\mathbf{u})\frac{\mathrm{d}t}{\mathrm{d}\tau} \tag{14}$$

where $\partial M/\partial \mathbf{x}$ is the Jacobian of the transformation [assumed to be nonsingular: namely, $\det(\partial M/\partial \mathbf{x}) \neq 0$]. The inverse transformation

$$\mathbf{x} = M^{-1}(\mathbf{y}) \tag{15}$$

$$\mathbf{u} = \alpha(M^{-1}(\mathbf{y})) + \beta(M^{-1}(\mathbf{y}))\mathbf{v} \tag{16}$$

provides the old state and control if the new ones are given. The original performance index (10) can be manipulated to yield [18]

$$J = \int_{\tau_0}^{\tau_f} T(\mathbf{y}, \mathbf{v})\frac{\mathrm{d}t}{\mathrm{d}\tau}\,\mathrm{d}\tau \tag{17}$$

where

$$T(\mathbf{y}, \mathbf{v}) = L(M^{-1}(\mathbf{y}), \alpha(M^{-1}(\mathbf{y})) + \beta(M^{-1}(\mathbf{y}))\mathbf{v}) \tag{18}$$

The new optimal control problem is stated by Eqs. (13) and (17), together with the two transformed boundary conditions that now read $\mathbf{y}(\tau_0) = \mathbf{y}_0$ and $\mathbf{y}(\tau_f) = \mathbf{y}_f$ (obtained by direct application of map (11) to the initial and final states). As proposed by Agrawal and Faiz [9], the necessary conditions of optimality can be solved for this new system, and the optimal trajectories of $\mathbf{y}(\tau)$ and $\mathbf{v}(\tau)$ can be computed. The old variables $\mathbf{x}(t)$ and $\mathbf{u}(t)$ can be derived by means of inverse transformations (15) and (16). Finally, by manipulating $t = t(\tau)$, the relation between the two independent variables can be derived [19]:

$$t - t_0 = \int_{\tau_0}^{\tau}\left(\frac{\mathrm{d}t}{\mathrm{d}\tau}\right)\mathrm{d}\tau \tag{19}$$

A. Linear Equations of Motion

The formulated linearizing transformation is now shown and applied to nonlinear dynamic system (6) ($n = 3$ and $m = 1$). The control is scalar; therefore, $\mathbf{u} = u$ and $\mathbf{v} = v$. The devised map for the states is

$$\mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} h/r - \mu/h \\ -v_r \\ \theta \end{bmatrix} = \mathbf{M}(\mathbf{x}) \tag{20}$$

whereas the devised map for transformation (12) is simply

$$u = hv \tag{21}$$

which means that $\alpha = 0$ and $\beta = h$. The Jacobian of Eq. (20) is

$$\frac{\partial \mathbf{M}}{\partial \mathbf{x}} = \begin{bmatrix} -h/r^2 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} \tag{22}$$

with $\det(\partial \mathbf{M}/\partial \mathbf{x}) = -h/r^2$, meaning that the transformation is nonsingular for bound trajectories ($0 < r \ll \infty$).

We now take into account a slightly modified version of transformations (20) and (21) by noticing that θ does not affect nonlinear system (4). The idea is to neglect the last row of map (20) and assume θ, a state of the old system, to be an independent variable in the new system: namely, $\tau = \theta$. Assuming the angle θ to be an independent variable is a common technique used to further reduce the order of differential system (4) (see [14,17,19]). Thus, we are only interested in the first two components of $\mathbf{y}$; therefore, from now on, we use the notation $\mathbf{y} = [y_1, y_2]^T$. Taking into account the conservation of angular momentum (2), the independent variable transformation is simply

$$\frac{\mathrm{d}t}{\mathrm{d}\tau} = \frac{\mathrm{d}t}{\mathrm{d}\theta} = \frac{r^2}{h}$$

and, by virtue of Eq. (14), the derivative $\mathbf{y}'$ can be written as

$$\mathbf{y}' = \frac{\partial M}{\partial \mathbf{x}}(\mathbf{f} + G\mathbf{u})\frac{\mathrm{d}t}{\mathrm{d}\theta} = \begin{bmatrix} -h/r^2 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix} \times \left(\begin{bmatrix} v_r \\ h/r^2 \\ h^2/r^3 - \mu/r^2 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1/r^2 \end{bmatrix} u \right) \frac{r^2}{h} = \begin{bmatrix} y_2 \\ -y_1 - v \end{bmatrix} \tag{23}$$

Enforcing $\mathbf{y}'$ to be produced by a linear system of the kind $\mathbf{y}' = A\mathbf{y} + B\mathbf{v}$, the characteristic matrices A and B of the new system turn out to be

$$\mathbf{y}' = A\mathbf{y} + B\mathbf{v} = \underbrace{\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}}_{A} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} + \underbrace{\begin{bmatrix} 0 \\ -1 \end{bmatrix}}_{B} v \tag{24}$$

Furthermore, manipulating Eq. (17), the performance index written in new variables reads

$$J = \int_{\theta_0}^{\theta_f} v^2 \, \mathrm{d}\theta \tag{25}$$

It is worth noticing that map (20) gives rise to linear system (24) and to quadratic objective function (25). This is important because, in agreement with Eq. (17), no conditions are imposed on the form of the new objective function. The two-point boundary conditions of the new problem are

$$\mathbf{y}(\theta_0) = [1/r_0 - 1, -v_{r,0}]^T$$

and

$$\mathbf{y}(\theta_f) = [1/r_f - 1, -v_{r,f}]^T$$

and θ_0 and θ_f are fixed.

The feedback control of a linear system supplemented by a quadratic performance index is a well-known problem in control theory. It is called linear-quadratic regulator and its solution relies on the matrix Riccati equation [2]. Following the method developed in [6,7], we address the solution of this problem by means of the generating-function method. This is an elegant approach that exploits the properties of the canonical transformations, defined in the frame of Hamiltonian systems, to solve the Hamilton–Jacobi–Bellmann equation of the feedback control problem. We discuss this technique in the next section.

IV. Solving the Linear-Quadratic Regulator via Generating Functions

The linear-quadratic regulator addresses the problem of minimizing a scalar performance index written in the form

$$J = \frac{1}{2}\int_{\tau_0}^{\tau_f} (\mathbf{y}^T Q \mathbf{y} + \mathbf{v}^T R \mathbf{v}) \, \mathrm{d}\tau \tag{26}$$

subject to the linear dynamics

$$\mathbf{y}' = A\mathbf{y} + B\mathbf{v} \tag{27}$$

where $\mathbf{y} \in \mathbb{R}^n$; $\mathbf{v} \in \mathbb{R}^m$; in general, $m \leq n$;A and B are $n \times n$ and $n \times m$ matrices, respectively; Q and R are two $n \times n$ and $m \times m$ matrices, respectively, positive semidefinite and positive definite matrices. The initial and final conditions are given by

$$\mathbf{y}(\tau_0) = \mathbf{y}_0, \qquad \mathbf{y}(\tau_f) = \mathbf{y}_f \tag{28}$$

and τ_0 and τ_f are fixed. The Hamiltonian of problems (26–28) is

$$H(\mathbf{y}, \lambda, \mathbf{v}) = \tfrac{1}{2}(\mathbf{y}^T Q \mathbf{y} + \mathbf{v}^T R \mathbf{v}) + \lambda^T (A\mathbf{y} + B\mathbf{v}) \tag{29}$$

where the set of costates, or Lagrange multipliers, $\lambda \in \mathbb{R}^n$ was introduced. From Pontryagin's principle [1], the optimal solution is an extremum of the Hamiltonian. This yields the necessary condition

$$\frac{\partial H}{\partial \mathbf{v}} = 0 \tag{30}$$

which, in our case, allows us to obtain an explicit expression of $\mathbf{v}$ in terms of the Lagrange multipliers

$$\mathbf{v} = -R^{-1} B^T \boldsymbol{\lambda} \tag{31}$$

Substituting Eq. (31) into Eq. (29), the Hamiltonian is

$$H(\mathbf{y}, \boldsymbol{\lambda}) = \frac{1}{2}\begin{bmatrix} \mathbf{y} \\ \boldsymbol{\lambda} \end{bmatrix}^T \begin{bmatrix} Q & A^T \\ A & -BR^{-1}B^T \end{bmatrix} \begin{bmatrix} \mathbf{y} \\ \boldsymbol{\lambda} \end{bmatrix} \tag{32}$$

and the dynamics of the states and costates reduce to

$$\begin{bmatrix} \mathbf{y}' \\ \boldsymbol{\lambda}' \end{bmatrix} = \begin{bmatrix} A & -BR^{-1}B^T \\ -Q & -A^T \end{bmatrix} \begin{bmatrix} \mathbf{y} \\ \boldsymbol{\lambda} \end{bmatrix} \tag{33}$$

To find the optimal guidance law, the Euler–Lagrange equations (33) have to be solved with initial and final conditions (28). The solution of system (33) is $[\mathbf{y}(\tau), \lambda(\tau)]^T$ ($\tau \in [\tau_0, \tau_f]$), which by means of Eq. (31) yields the optimal guidance law $\mathbf{v}(\tau)$ ($\tau \in [\tau_0, \tau_f]$).

Equation (33), supplemented by conditions (28), represents the classic two-point boundary-value problem derived by the optimal control theory. In this case, the problem is linear and so the solution is analytical. For nonlinear problems, any change in the boundary conditions would require a new solution of the two-point boundary-value problem. In the following, we show that the generic state can be embedded in the solution of Eq. (33) in an analytical fashion. In this way, the optimal solution is an analytic function of the state, which is the essence of the optimal feedback control problem.

A. Generating-Function Method

The generating-function method for the solution of two-point boundary-value problems is reported next. This technique exploits fundamental links between optimal control theory and Hamiltonian dynamics. For a detailed derivation of the method, the reader can refer to the works of Park et al. [6,7].

The idea of this method is to exploit the properties of the generating functions associated with the transformations between a fixed state $(\mathbf{y}_0, \boldsymbol{\lambda}_0, \tau_0)$ and a moving state $(\mathbf{y}, \boldsymbol{\lambda}, \tau)$. These two states coincide at $\tau = \tau_0$, and therefore the generating functions must define an identity transformation at $\tau = \tau_0$. This means that among the four possible forms of generating function [7], the choice is restricted to only those two that are functions of both coordinates and momenta.

Suppose that we have a generating function $F_2(\mathbf{y}, \boldsymbol{\lambda}_0, \tau, \tau_0)$. Because Hamiltonian (32) is quadratic, F_2 can be put in a quadratic form [7]:

$$F_2(\mathbf{y}, \boldsymbol{\lambda}_0, \tau, \tau_0) = \frac{1}{2}\begin{bmatrix} \mathbf{y} \\ \boldsymbol{\lambda}_0 \end{bmatrix}^T \begin{bmatrix} F_{yy}(\tau, \tau_0) & F_{y\lambda_0}(\tau, \tau_0) \\ F_{\lambda_0 y}(\tau, \tau_0) & F_{\lambda_0\lambda_0}(\tau, \tau_0) \end{bmatrix} \begin{bmatrix} \mathbf{y} \\ \boldsymbol{\lambda}_0 \end{bmatrix} \tag{34}$$

The function F_2 satisfies the Hamilton–Jacobi equation for the generating functions and it can therefore be used to find the unknown boundary conditions using the given boundary conditions. In particular, from the properties of F_2, we have

$$\boldsymbol{\lambda} = \frac{\partial F_2}{\partial \mathbf{y}} = [F_{yy} \quad F_{y\lambda_0}] \begin{bmatrix} \mathbf{y} \\ \boldsymbol{\lambda}_0 \end{bmatrix} \tag{35}$$

Hamiltonian (32) can be expressed as a function of $\mathbf{y}$ and $\boldsymbol{\lambda}_0$ by using Eq. (35):

$$H = \frac{1}{2}\begin{bmatrix}\mathbf{y}\\ \boldsymbol{\lambda}_0\end{bmatrix}^T \begin{bmatrix} I & F_{yy}\\ 0 & F_{\lambda_0 y}\end{bmatrix}\begin{bmatrix} Q & A^T\\ A & -BR^{-1}B^T\end{bmatrix}\begin{bmatrix} I & 0\\ F_{yy} & F_{y\lambda_0}\end{bmatrix}\begin{bmatrix}\mathbf{y}\\ \boldsymbol{\lambda}_0\end{bmatrix} \tag{36}$$

Because the Hamiltonian of the fixed state can be assumed to be zero without any loss of generality [6], then the Hamiltonian of the moving state and the generating function satisfies the Hamilton–Jacobi partial differential equation $\partial F_2/\partial\tau + H = 0$; namely,

$$\begin{bmatrix}\mathbf{y}\\ \boldsymbol{\lambda}_0\end{bmatrix}^T \left(\begin{bmatrix} F'_{yy} & F'_{y\lambda_0}\\ F'_{\lambda_0 y} & F'_{\lambda_0\lambda_0}\end{bmatrix} + \begin{bmatrix} I & F_{yy}\\ 0 & F_{\lambda_0 y}\end{bmatrix}\begin{bmatrix} Q & A^T\\ A & -BR^{-1}B^T\end{bmatrix}\begin{bmatrix} I & 0\\ F_{yy} & F_{y\lambda_0}\end{bmatrix}\right)\begin{bmatrix}\mathbf{y}\\ \boldsymbol{\lambda}_0\end{bmatrix} = 0 \tag{37}$$

From Eq. (37), it is possible to extract the matrix Riccati equations for the submatrix components of the generating function

$$\begin{aligned} &F'_{yy} + Q + F_{yy}A + A^T F_{yy} - F_{yy}BR^{-1}B^T F_{yy} = 0,\\ &F'_{y\lambda_0} + A^T F_{y\lambda_0} - F_{yy}BR^{-1}B^T F_{y\lambda_0} = 0,\\ &F'_{\lambda_0\lambda_0} - F_{\lambda_0 y}BR^{-1}B^T F_{y\lambda_0} = 0 \end{aligned} \tag{38}$$

The initial conditions for Eqs. (38),

$$F_{yy}(\tau_0,\tau_0) = 0_{n\times n}, \quad F_{y\lambda_0}(\tau_0,\tau_0) = I_{n\times n}, \quad F_{\lambda_0\lambda_0}(\tau_0,\tau_0) = 0_{n\times n} \tag{39}$$

are taken from the identity transformation

$$F_2(\mathbf{y},\boldsymbol{\lambda}_0,\tau=\tau_0,\tau_0) = \mathbf{y}^T\boldsymbol{\lambda}_0$$

which verifies the identity at $\tau = \tau_0$. The set of matrix ordinary differential equations (38) can be integrated with initial conditions (39); this procedure yields the generating function F_2 and therefore, through Eq. (35), the function

$$\boldsymbol{\lambda} = \boldsymbol{\lambda}(\mathbf{y},\boldsymbol{\lambda}_0,\tau,\tau_0)$$

Nevertheless, the stated problem is a hard-constraint problem (the initial and final states are both fixed), and so it would be useful to have

$$\boldsymbol{\lambda}_0 = \boldsymbol{\lambda}_0(\mathbf{y}_0,\mathbf{y}_f,\tau_0,\tau_f)$$

This function can be obtained by (see [7]):

$$\mathbf{y}_0 = \frac{\partial F_2}{\partial\boldsymbol{\lambda}_0} = F_{\lambda_0 y}\mathbf{y}_f + F_{\lambda_0\lambda_0}\boldsymbol{\lambda}_0 \tag{40}$$

Equation (40) can be used to extract the required initial Lagrange multiplier:

$$\boldsymbol{\lambda}_0 = \boldsymbol{\lambda}_0(\mathbf{y}_0,\mathbf{y}_f,\tau_0,\tau_f) = F^{-1}_{\lambda_0\lambda_0}(\tau_f,\tau_0)(\mathbf{y}_0 - F_{\lambda_0 y}(\tau_f,\tau_0)\mathbf{y}_f) \tag{41}$$

This condition determines the initial costate as a function of the given initial and final states; therefore, through Eq. (31), the initial value of the control is

$$\mathbf{v}_0(\mathbf{y}_0,\mathbf{y}_f,\tau_0,\tau_f) = -R^{-1}B^T\boldsymbol{\lambda}_0(\mathbf{y}_0,\mathbf{y}_f,\tau_0,\tau_f) \tag{42}$$

In addition, relation (41) is valid for any time $\tau \geq \tau_0$ and with generic state $\mathbf{y}$

$$\boldsymbol{\lambda}(\mathbf{y},\tau) = F^{-1}_{\lambda_0\lambda_0}(\tau_f,\tau)(\mathbf{y} - F_{\lambda_0 y}(\tau_f,\tau)\mathbf{y}_f) \tag{43}$$

Thus, the optimal feedback guidance law is

$$\mathbf{v}(\mathbf{y},\tau) = -R^{-1}B^T\boldsymbol{\lambda}(\mathbf{y},\tau) \tag{44}$$

where the dependence on $\mathbf{y}_f$ and τ_f was suppressed because they are both fixed in the current problem. The computation of $\boldsymbol{\lambda}(\mathbf{y},\tau)$ involves the inversion of the submatrix $F_{\lambda_0\lambda_0}$; as a result, the solution is singular when $\det(F_{\lambda_0\lambda_0}) = 0$.

V. Optimal Feedback Low-Thrust Transfers

The optimal low-thrust problem with modulated inverse-square-distance radial thrust was stated through Eqs. (5–8). The linearizing transformation was applied to this problem, and linear state-space representation (24), supplemented by quadratic objective function (25), was derived. To obtain optimal feedback solutions, this linear-quadratic regulator problem was solved using the generating-function method. In this section, we first solve problems (24) and (25), and then we transform the solution back into the original variables. The solution is explained with the aid of sample cases.

The two-point conditions (8) in the new variables read

$$\begin{aligned} \mathbf{y}(\tau_0) &= \begin{bmatrix} y_{1,0}\\ y_{2,0}\end{bmatrix} = \begin{bmatrix} h/r_0 - \mu/h\\ -v_{r,0}\end{bmatrix},\\ \mathbf{y}(\tau_f) &= \begin{bmatrix} y_{1,f}\\ y_{2,f}\end{bmatrix} = \begin{bmatrix} h/r_f - \mu/h\\ -v_{r,f}\end{bmatrix} \end{aligned} \tag{45}$$

By comparing objective functions (25) and (26), we find that $Q = 0_{2\times 2}$ and $R = 2$; moreover, substituting A and B given by Eq. (24), Eq. (33) becomes

$$\begin{bmatrix}\mathbf{y}'\\ \boldsymbol{\lambda}'\end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0\\ -1 & 0 & 0 & -1/2\\ 0 & 0 & 0 & 1\\ 0 & 0 & -1 & 0\end{bmatrix}\begin{bmatrix}\mathbf{y}\\ \boldsymbol{\lambda}\end{bmatrix} \tag{46}$$

Matrix ordinary differential equation (38) can be integrated with initial conditions (39). The analytical solution of the submatrices $F_{y\lambda_0}$ and $F_{\lambda_0\lambda_0}$ involved in Eq. (43) is

$$F_{y\lambda_0}(\theta,\theta_0) = F^T_{\lambda_0 y}(\theta_0,\theta) = \begin{bmatrix} \cos(\theta-\theta_0) & \sin(\theta-\theta_0)\\ -\sin(\theta-\theta_0) & \cos(\theta-\theta_0)\end{bmatrix} \tag{47}$$

$$F_{\lambda_0\lambda_0}(\theta,\theta_0) = \begin{bmatrix} (\theta-\theta_0)/4 - (\sin 2(\theta-\theta_0))/8 & -(\sin^2(\theta-\theta_0))/4\\ (\sin^2(\theta-\theta_0))/4 & (\theta-\theta_0)/4 - (\sin 2(\theta-\theta_0))/8\end{bmatrix} \tag{48}$$

It is worth observing that $\det(F_{\lambda_0\lambda_0}(\theta_f,\theta_0)) = 0$ when $\theta_f - \theta_0 = 0$. In this case, the feedback gains of the optimal control law would tend to infinity, because θ represents the independent variable. From Eq. (41), we obtain the initial Lagrange multiplier:

$$\begin{aligned} \lambda_{1,0}(\mathbf{y}_0,\mathbf{y}_f,\theta_0,\theta_f) &= 4b_1^{-1}(2a_2 s_f^2 + a_1 b_2),\\ \lambda_{2,0}(\mathbf{y}_0,\mathbf{y}_f,\theta_0,\theta_f) &= 4b_1^{-1}(2a_1 s_f^2 - a_2 b_3) \end{aligned} \tag{49}$$

where

$$\begin{aligned} a_1(\mathbf{y}_0,\mathbf{y}_f,\theta_0,\theta_f) &= y_{1,0} - y_{1,f}c_f + y_{2,f}s_f,\\ a_2(\mathbf{y}_0,\mathbf{y}_f,\theta_0,\theta_f) &= y_{2,0} - y_{1,f}s_f - y_{2,f}c_f,\\ b_1(\mathbf{y}_0,\mathbf{y}_f,\theta_0,\theta_f) &= 2s_f^4 + 2s_f^2c_f^2 - 2\Delta\theta_f^2,\\ b_2(\mathbf{y}_0,\mathbf{y}_f,\theta_0,\theta_f) &= -2s_f c_f - 2\Delta\theta_f,\\ b_3(\mathbf{y}_0,\mathbf{y}_f,\theta_0,\theta_f) &= -2s_f c_f + 2\Delta\theta_f \end{aligned} \tag{50}$$

and, for brevity, $s_f = \sin(\theta_f - \theta_0)$, $c_f = \cos(\theta_f - \theta_0)$, and $\Delta\theta_f = \theta_f - \theta_0$. Finally, by integrating Eq. (46), the optimal feedback solution for the linearized problem can be achieved

$$y_1(\mathbf{y}_0,\mathbf{y}_f,\theta_0,\theta_f,\theta) = y_{1,0}c + y_{2,0}s + b_1^{-1}[(2a_2s_f^2 + a_1b_2)(s - \Delta\theta c) - (2a_1s_f^2 - a_2b_3)\Delta\theta s] \tag{51}$$

$$y_2(\mathbf{y}_0,\mathbf{y}_f,\theta_0,\theta_f,\theta) = y_{2,0}c - y_{1,0}s + b_1^{-1}[(2a_2s_f^2 + a_1b_2)\Delta\theta s + (2a_1s_f^2 - a_2b_3)(s + \Delta\theta c)] \tag{52}$$

together with the Lagrange multipliers

$$\lambda_1(\mathbf{y}_0, \mathbf{y}_f, \theta_0, \theta_f, \theta) = 4b_1^{-1}[(2a_2 s_f^2 + a_1 b_2)c + (2a_1 s_f^2 - a_2 b_3)s] \quad (53)$$

$$\lambda_2(\mathbf{y}_0, \mathbf{y}_f, \theta_0, \theta_f, \theta) = 4b_1^{-1}[(2a_1 s_f^2 - a_2 b_3)c - (2a_2 s_f^2 + a_1 b_2)s] \quad (54)$$

where $s = \sin(\theta - \theta_0)$, $c = \cos(\theta - \theta_0)$, and $\Delta\theta = \theta - \theta_0$. It can be noted that the solution is singular for $b_1 = 0$, which occurs again for $\theta_f - \theta_0 = 0$.

The optimal guidance law for the linear-quadratic regulator, $\mathbf{v} = \mathbf{v}(\mathbf{y}_0, \mathbf{y}_f, \tau_0, \tau)$, can be obtained through Eq. (31). This solution is valid for any $\theta \leq \theta_f$; therefore, by means of Eq. (44), the optimal feedback control $\mathbf{v} = \mathbf{v}(\mathbf{y}, \tau)$ is derived. In general, the optimal feedback solution is derived from the open-loop optimal solution by replacing ($\mathbf{y}_0$, τ_0) with ($\mathbf{y}$, τ). The trajectory $\mathbf{y}(\theta)$ ($\theta \in [\theta_0, \theta_f]$), described by Eqs. (51) and (52), is now transformed back into the form that uses the variables of the original problem.

A. Inverse Transformation

The inverse transformation is now used to derive the solution of the original problem in its final form. Equation (20) yields $r = h^2/(hy_1 + \mu)$ and $v_r = -y_2$; therefore, the optimal feedback trajectory of the transfer problem is

$$r(r_0, v_{r,0}, r_f, v_{r,f}, \theta_0, \theta_f, \theta) = h^2/(\mu + hy_1(r_0, v_{r,0}, r_f, v_{r,f}, \theta_0, \theta_f, \theta)) \quad (55)$$

$$v_r(r_0, v_{r,0}, r_f, v_{r,f}, \theta_0, \theta_f, \theta) = -y_2(r_0, v_{r,0}, r_f, v_{r,f}, \theta_0, \theta_f, \theta) \quad (56)$$

where the boundary conditions were embedded using relations (45), and y_1 and y_2 are expressed by Eqs. (51) and (52). The optimal guidance using Eqs. (21) and (44) reads

$$u(r_0, v_{r,0}, r_f, v_{r,f}, \theta_0, \theta_f, \theta) = hv = h/2\lambda_2(r_0, v_{r,0}, r_f, v_{r,f}, \theta_0, \theta_f, \theta) \quad (57)$$

where the function λ_2 is given in Eq. (54). The time of flight can be found by quadrature through

$$t_f - t_0 = \int_{\theta_0}^{\theta_f} r^2(\theta)\,\mathrm{d}\theta \quad (58)$$

The optimal feedback solution in terms of trajectory and control is derived from Eqs. (55–57) by replacing the initial values of $[r_0, \theta_0, v_{r,0}]$ with the corresponding current values $[r, \theta, v_r]$.

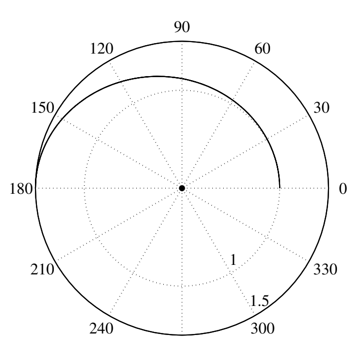
B. Optimal Orbital Transfers

The optimal feedback control technique developed so far is suitable for radially accelerated orbital transfers with modulated inverse-square-distance radial thrust. This kind of thrust changes the spacecraft's energy while conserving the angular momentum; therefore, transfers between circular orbits are forbidden. In this section, we apply solutions (55–58) to derive optimal transfers between the Earth's orbit and an elliptical orbit with the apoapsis on the Mars' orbit. We first derive the nominal open-loop guidance laws for three sample cases with different values of $\theta_f - \theta_0$ (corresponding to different numbers of revolutions around the sun). In the next subsection, we derive the optimal feedback solution for a sample case.

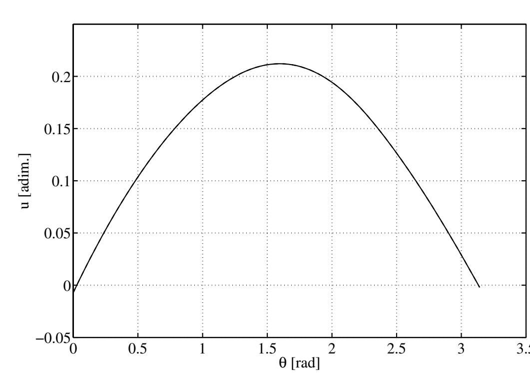
Without any loss of generality, we consider normalized variables. The radius of the Earth's orbit, the velocity of the Earth on its circular orbit, and the angular velocity of the Earth around the sun are all set to one (with these coordinates, it turns out that $\mu = h = 1$). The initial conditions are $r_0 = 1$ and $v_{r,0} = 0$, and the final conditions are $r_f = 1.5$ and $v_{r,f} = 0$; we fix $\theta_0 = 0$ and choose three different values for θ_f for the cases considered. In this way, for each value of θ_f, we obtain a nominal open-loop solution. It is worth remarking that each solution is obtained by only evaluating Eqs. (55–57).

We study three different cases: a, b, and c, with θ_f equal to π, 2π, and 4π, respectively. For each case, both the nominal transfer trajectory and the guidance law are plotted (see Figs. 1–3). The transfer times for the three cases obtained by numerically integrating Eq. (58) are 259.8, 397.6, and 780.0 days, respectively.

For the sake of completeness, the analytical feedback solutions were compared with those found by a standard open-loop optimizer. A numerical scheme was implemented for the solution of original problems (5–8). This is a direct shooting algorithm that computes the optimal values of the control function at given mesh points: namely, u_i ($i = 1, \ldots, n_M$, where n_M is the number of mesh points). The optimal control law $u(t)$ ($t \in [t_0, t_f]$) is approximated by means of cubic spline interpolation. It is remarkable that in the three cases shown, the numerical open-loop solution almost overlaps the feedback analytical solution found by globally linearization of the dynamics and application of the generating-function method. In particular, for case a, four mesh points suffice to derive a numerical solution close to the analytical solution. For cases b and c, 8 and 12 mesh points were used, respectively. The initial guesses $u_i = 0$ and $i = 1, \ldots, n_M$ were adopted. Apparently, the optimal solution of the linear-quadratic regulator, transformed back into the old coordinates, corresponds to the optimal solution of original problems (5–8). Although the numerical open-loop results approximate the analytical feedback solutions, they are not dependent on the current state. Thus, for arbitrary small changes in the initial conditions, the numerical solutions need to be computed again, because the nominal guidance found is no longer valid. This concept is clearly shown in the following subsection.

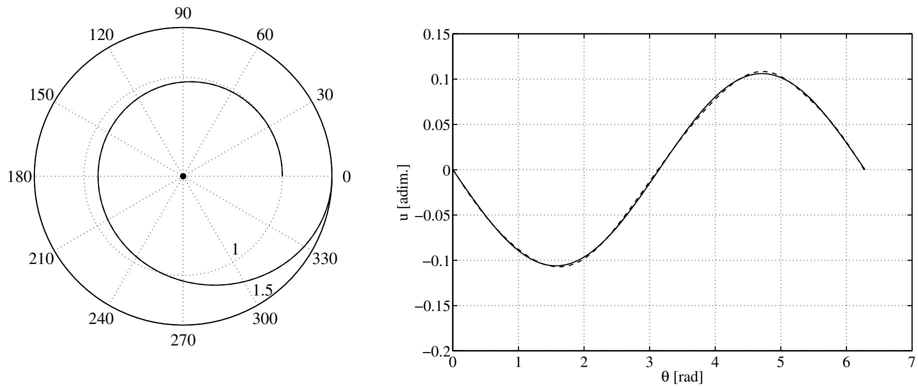


a) Transfer trajectory



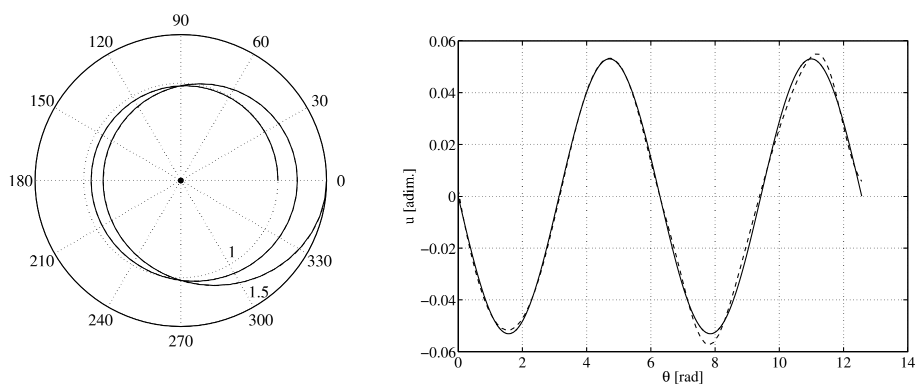
b) Control law

Fig. 1 Optimal solution for case a ($\theta_f = \pi$). Both the analytical feedback (solid lines) and the numerical open-loop solution (dashed lines) obtained with $n_M = 4$ are shown. (No difference can be appreciated between the curves because the two solutions are perfectly overlapped in this case.)



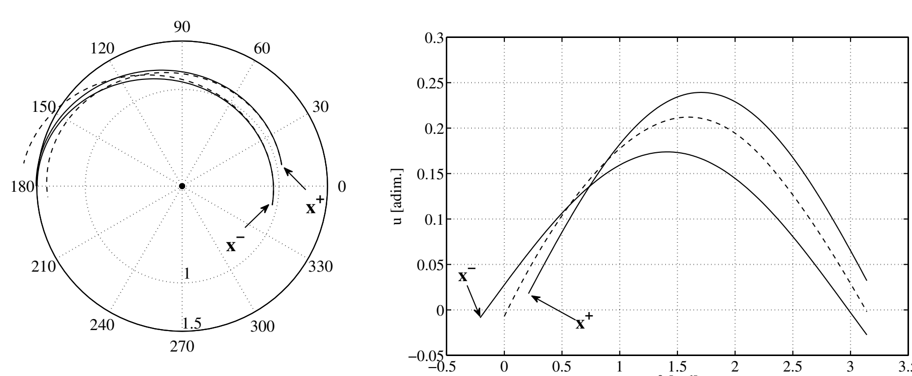
a) Transfer trajectory **b) Control law**

Fig. 2 Optimal solution for case b ($\theta_f = 2\pi$). Both the analytical feedback (solid lines) and the numerical open-loop solution (dashed lines) obtained with $n_M = 8$ are shown.



a) Transfer trajectory **b) Control law**

Fig. 3 Optimal solution for case a ($\theta_f = 4\pi$). Both the analytical feedback (solid lines) and the numerical open-loop solution (dashed lines) obtained with $n_M = 12$ are shown.



a) Transfer trajectories **b) Control laws**

Fig. 4 Solutions associated with two perturbed initial conditions (x^- and x^+) of case a ($\theta_f = \pi$). The analytical feedback solution (solid lines) is optimal in relation to these new initial conditions. This solution respects the final conditions. If the nominal open-loop control law (dashed lines) is applied to the new initial conditions, the solution does not respect the final conditions and the spacecraft does not target the desired state.

C. Optimal Feedback Orbital Transfers

Nominal solutions were obtained with fixed initial and final states $\mathbf{x}_0 = [r_0, \theta_0, v_{r,0}]^T$ and $\mathbf{x}_f = [r_f, \theta_f, v_{r,f}]^T$, respectively. In this section, we consider perturbed initial conditions and show that new optimal solutions are automatically provided by Eqs. (55–58) in a totally analytical fashion.

We consider generic states

$$\mathbf{x} = [r_0 + \delta r_0, \theta_0 + \delta\theta_0, v_{r,0} + \delta v_{r,0}]^T$$

with

$$|\delta r_0| \le \delta r_0^{\max}, \qquad |\delta\theta_0| \le \delta\theta_0^{\max}, \qquad |\delta v_{r,0}| \le \delta v_{r,0}^{\max}$$

defined inside a cube centered at $\mathbf{x}_0$. Given any new initial state $\mathbf{x}$, it is possible to extract the new optimal solution by evaluation of Eqs. (55–58) with $r = r_0 + \delta r_0$, $\theta = \theta_0 + \delta\theta_0$, and $v_r = v_{r,0} + \delta v_{r,0}$ in place of r_0, θ_0, and $v_{r,0}$, respectively.

To verify the validity of the solution derived, we assume

$$\delta r_0^{\max} = 0.05, \qquad \delta\theta_0^{\max} = \pi/15, \qquad \delta v_{r,0}^{\max} = 0.05$$

and guess random states inside this box. For each initial condition and for the preceding three cases, the problem solution allows us to extract the new optimal transfer trajectory from $\mathbf{x}$ to $\mathbf{x}_f$. In Fig. 4, we show the new optimal solution in terms of transfer trajectory and control profile, associated with

$$\mathbf{x}^- = [r_0 - \delta r_0^{\max}, \theta_0 - \delta\theta_0^{\max}, v_{r,0} - \delta v_{r,0}^{\max}]^T$$

and

$$\mathbf{x}_0^+ = [r_0 + \delta r_0^{\max}, \theta_0 + \delta\theta_0^{\max}, v_{r,0} + \delta v_{r,0}^{\max}]^T$$

for case a. In the same figure, the numerical open-loop optimal control law illustrated in Fig. 1a and reported in Fig. 4b (dashed lines) was applied to derive solutions stating from the perturbed states $\mathbf{x}^-$ and $\mathbf{x}^+$. As can be seen in Fig. 4a (dashed lines), the open-loop control law fails with perturbed initial conditions, because the resulting orbit does not respect the final conditions. In this case, the solution of another optimal control problem is required. This is avoided with the analytical feedback solution derived in this paper.

VI. Conclusions

An analytical solution to the optimal feedback control problem in orbital transfers with modulated inverse-square-distance radial thrust was derived. The nonlinear problem was transformed into a classic linear-quadratic regulator by application of a suitably devised diffeomorphic transformation. This kind of map totally preserves the model accuracy, because no remainder truncation is involved with the process. Once the problem is formulated by means of linear dynamics and a quadratic objective function, the solution of the optimal feedback control problem can rely on well-known methods. In this paper, we applied the elegant generating-function method that exploits fundamental links between Hamiltonian dynamics and optimal control theory. Once the linear-quadratic regulator was solved, the optimal feedback solution was simply obtained by an inverse transformation. We showed the significant value of the proposed approach through sample cases. First, optimal orbital transfers were defined, then by perturbing the initial state, a family of new optimal transfers related to such new initial conditions was obtained by simple function evaluations.

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